5. GRAPHS ON SURFACES

§5.1. Graphs

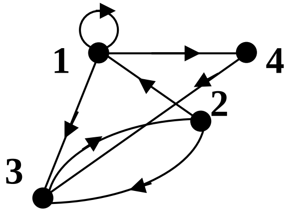
A graph, by itself, is a combinatorial object rather than a topological one. But when we relate a graph to a surface through the process of 'embedding' we move into the realm of topology.

A **graph** is a set, X, of elements together with a relation on X. We often draw pictures of graphs, where the elements of X are represented by dots and the relation by a set of arrows connecting certain dots to others. The elements of X are called **vertices** and the arrows joining vertices related by the relation are called **edges**.

Example 1:
The following is a graph on the set {1, 2, 3, 4}:

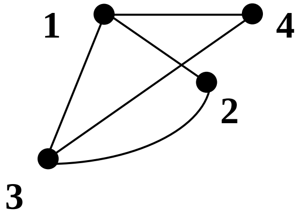
{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (3, 2), (4, 3)}.

These are the (directed) edges and they can be represented by the following diagram. There are 4 vertices and 7 edges.



However here we’ll only consider undirected graphs with no loops. A **loop** is an edge (V, V) from a vertex to itself. An **undirected graph** is one where the relation is symmetric, that is, if U is related to V then V is related to U. So if there’s an arrow in one direction there’s always one in the opposite direction. In an undirected graph there’s no need to use arrows since we know that the relation goes in both directions, and so we simply use edges without arrows.

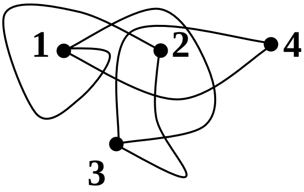
Example 2: The following is an undirected graph with no loops.

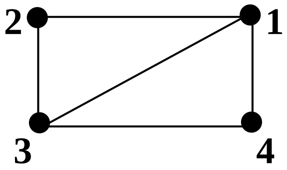


From now on, when we use the word ‘graph’, we’ll mean that it’s undirected and has no loops.

A graph is a combinatorial structure where the only consideration is which vertices are adjacent to which. When we draw a graph the positions of the points representing the vertices are arbitrary. So are the routes of the edges. The edges needn’t be straight – they’re allowed to cross over other edges, and they could even wind around in more complicated ways. However we usually draw a graph in such a way that it gives as simple a picture as possible.

Example 3: The graph in example 2 could be redrawn as in the diagram on the left, but would look much better when drawn as the one on the right.





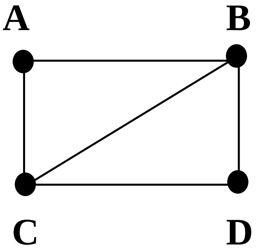
Notice that in the above example it's possible to draw the graph without any of the edges crossing. This isn't always possible, and we'll be very much concerned with the problem of when it is possible and when it isn't.

In a graph we say that two vertices are **adjacent** if there's an edge between them.

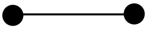
Example 4: In example 3, vertices 1 and 2 are adjacent but 2 and 4 are not.

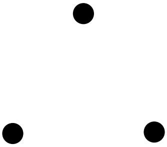
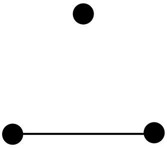
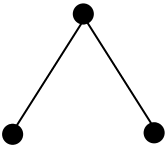
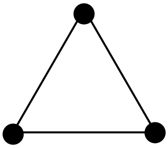
Two graphs X and Y are **equivalent** if there's a 1-1 and onto map $f: X \to Y$ such that V_1 and V_2 are adjacent in X if and only if $f(V_1)$ and $f(V_2)$ are adjacent in Y.

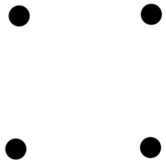
Example 5: The two graphs in example 3 are equivalent. And both are equivalent to the following graph. (It's essentially the same graph but with the vertices labelled differently.)



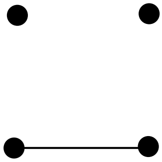
Example 6: The following list contains all the graphs with 4 vertices or less. Every graph with up to 5 vertices is equivalent to exactly one of these. They’ve been systematically classified according to the number of vertices, V, and edges, E.

 V = 1 E = 0	 V = 2 E = 0	 V = 2 E = 1

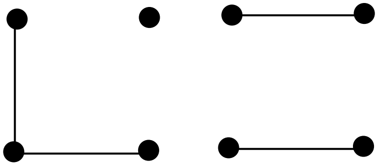
 V = 3 E = 0	 V = 3 E = 1	 V = 3 E = 2	 V = 3 E = 3



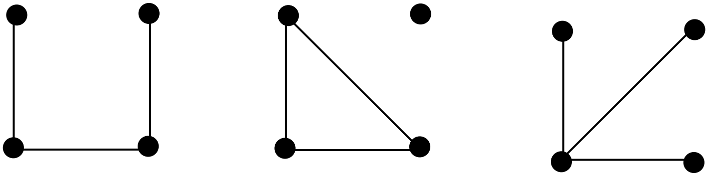
$V = 4$
$E = 0$



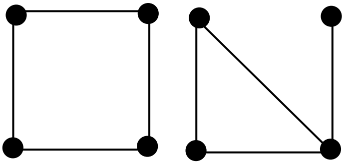
$V = 4$
$E = 1$



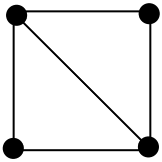
$V = 4$
$E = 2$



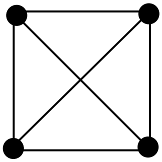
$V = 4$
$E = 3$



$V = 4$
$E = 4$



$V = 4$
$E = 5$

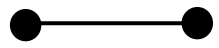


$V = 4$
$E = 6$

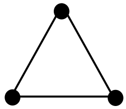
The **complete graph** on n vertices, denoted by $\mathbf{K}_n$, is the graph where every vertex is adjacent to every other vertex. The number of edges in $\mathbf{K}_n$ is clearly the binomial coefficient $\binom{n}{2}$.

Example 7: The following are the complete graphs on 6 vertices or less.

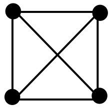
$\mathbf{K}_1$



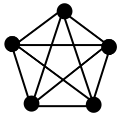
$\mathbf{K}_2$



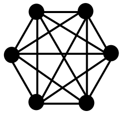
$\mathbf{K}_3$



$\mathbf{K}_4$



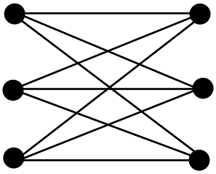
$\mathbf{K}_5$



$\mathbf{K}_6$

Another important family of graphs consists of the graphs $\mathbf{K}_{m,n}$ for various values of m and n (they don't have a name, just a symbol). The graph $\mathbf{K}_{m,n}$ has $m + n$ vertices divided into two subsets, one of size m and the other of size n. Every vertex in one subset is adjacent to every vertex in the other, but there are no edges connecting two vertices within the same subset.

Example 8: The following is $\mathbf{K}_{3,3}$:



This graph was once featured in an Air New Zealand advertisement, where the 6 vertices consisted of the cities Brisbane, Sydney, Melbourne, Auckland, Wellington and Christchurch. The edges represented the trans-Tasman routes.

§5.2. The Utilities Puzzle

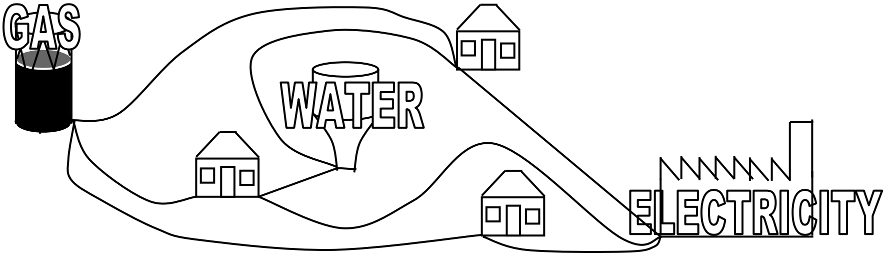
A graph has vertices and edges. So has a map, so what's the difference? Maps have faces, while graphs don't. It's the existence of faces on a map that give it its topological significance. Consider the following famous puzzle, called the Utilities Puzzle.

You have three houses and three 'utilities'. The utilities are a gasworks, a power station and a water reservoir. They have to pump gas, electricity and water to each of the three houses. But they have to do this so that the pipes and cables don't intersect.

You see this is a 2-dimensional problem. In real life (3-dimensional) there's no problem at all. Gas pipes can be routed under electrical cables or over water pipes. But we have to solve the puzzle in 2 dimensions.

The problem is to draw the three houses and three utilities on a sheet of paper, and draw lines to represent the pipes and cables, in such a way that they only meet one another at endpoints. You might like to have a go at this problem.

Example 9: The following is a near solution. Clearly we can't put in the remaining water pipe without crossing the lines we've drawn already.



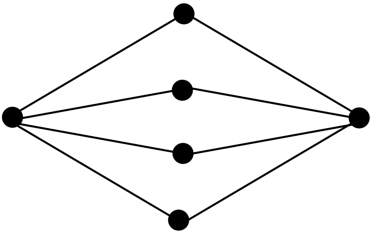
But that doesn't prove that the problem has no solution. Perhaps you can think of better places to put the six vertices or perhaps you can route the first eight edges in a really clever way so that the last one can be drawn without crossing the others.

Don't spend too long on this puzzle, because it's impossible! If you've tried to solve it for a few minutes you'll come to this conclusion, though you won't have a proof. But beware! Haven't you ever attempted puzzles where, after trying in vain for many minutes, you become quite convinced that it's impossible, only to have someone come along and show you a really clever solution? Not in this case, though. We're going to *prove* that this puzzle is impossible!

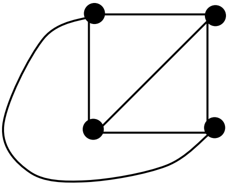
In the language of graph theory the network of pipes and cables is the graph $\mathbf{K}_{3,3}$. We want to draw this graph in the plane so that edges meet only at vertices. Or, to use a new concept that we will soon define, the problem is to *embed* $\mathbf{K}_{3,3}$ on a disk.

§5.3. Planarity

A graph is defined to be **planar** if it can be embedded in a disk. So $\mathbf{K}_{3,3}$ isn't planar. But $\mathbf{K}_{4,2}$ is:



So is $\mathbf{K}_4$:



Since a disk can be cut out of a sphere, any planar graph can be embedded in a sphere. On the other hand, if we can embed a graph in a sphere, we can cut out a small hole in the middle of one of the faces and we have an embedding of the graph in a disk. (Remember that a disk is homeomorphic to a sphere with one hole.)

So a graph is planar if and only if it can be embedded in a sphere. This is useful because often a sphere is more convenient to work with.

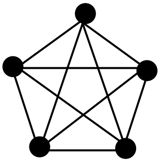
To prove that a graph is planar we can simply draw it, with edges not crossing. But how do we show that a graph, such as, $\mathbf{K}_{3,3}$ or $\mathbf{K}_5$ is *not* planar? The technique discussed here is to work out the average number of edges per face and compare this to the smallest number of edges for any face. But wait a minute. Graphs don't have faces!

That's true, but a graph embedded in a surface becomes a map, and maps have faces. So we *suppose* that the graph is planar, that is, it can be embedded in a sphere. But how can we count the number of faces if we're only *supposing* that the graph can be embedded? The answer is to use Euler's formula:

$$V + F - E = \chi$$

For planarity we use $\chi = 2$, the Euler characteristic of the sphere. Why not $\chi = 1$ for the disk? The answer is that we'll be assuming that there's a face on both sides of each edge of the map. If we have boundaries this will not be so.

Example 10: $\mathbf{K}_5$ is not planar.



Proof: For $\mathbf{K}_5$ we have $V = 5$ and $E = 10$.

Suppose that $\mathbf{K}_5$ is planar. Then embedding it in a sphere we can deduce that the number of faces must be:

$$F = 2 + E - V = 7.$$

The average number of edges per face must therefore be

$$\frac{2E}{F} = \frac{20}{7} = 2\tfrac{6}{7}.$$

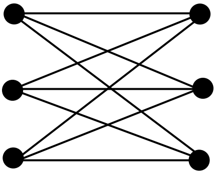
Why $\frac{2E}{F}$ and not just $\frac{E}{F}$? The reason is that every edge is associated with two faces – one on each side. So if you were to split each edge lengthwise, in such a way that each half edge was associated with only one face, you'd have 2E half edges to share among the F faces.

Now we wanted to prove that $\mathbf{K}_5$ can't be embedded in a sphere and we started out by supposing that it can be. We're clearly looking for a contradiction. So what's contradictory about the average number of edges per face being $2\tfrac{6}{7}$?

What's wrong is that it's less than 3. Every face must be surrounded by at least 3 edges (A face bounded by 2 edges would require that the two edges connect the same two vertices, and a face bounded by just 1 edge would mean that the graph has a loop.)

Now the average of a collection of numbers can't be less than the smallest of them. So here we have our contradiction!

Example 11: $\mathbf{K}_{3,3}$ is not planar.



Proof: Here V = 6 and E = 9.
Suppose that $\mathbf{K}_{3,3}$ can be embedded in a sphere.
The resulting map would have to have F faces where

$$6 + F - 9 = 2,$$

that is, it must have 5 faces.

The average number of edges per face would therefore be $\frac{18}{5} = 3\frac{3}{5}$.

This isn't less than 3, so where's the contradiction? The contradiction is that it's less than 4. You see, in this graph there are no cycles of length 3. Each edge takes you from one set of vertices to the other. Going along another edge must take you back to a different vertex in the first set. The smallest cycles in this graph therefore have length 4. The boundary of a face must be a cycle in the graph. So the smallest number of edges for each face is 4. But the average of these numbers is less than 4. This can't be, and so we have our contradiction.

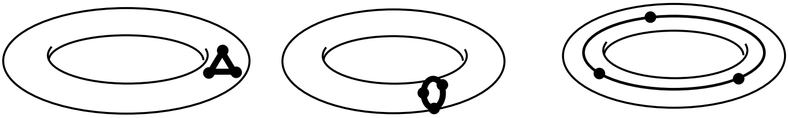
The **girth** of a graph is the length of the shortest cycle. The girth of $\mathbf{K}_5$ is 3 but the girth of $\mathbf{K}_{3,3}$ is 4. We get a contradiction if the average number of edges per face is less than the girth.

§5.4. Embedding a Graph in a Surface.

A graph can be **embedded** in a surface if it can be drawn on the surface in such a way that edges only meet at vertices (that is, they do not intersect each other).

On a sphere this results in a map, with faces surrounded by edges and each of these faces is homeomorphic to a disk. It is this fact that underlies our discussion of planarity.

But on other surfaces there are complications. We can draw a triangle ($\mathbf{K}_3$) on a torus. But fundamentally there are three ways of doing this.



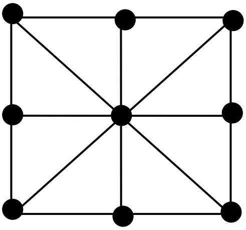
In the first case the triangle separates the torus into two pieces – a disk and a torus with a hole. In the second case there is no interior to the ‘triangle’ (which looks more like a circle). If the triangle is removed, it leaves a cylinder. In the third case, again there is no interior. Removing the ‘triangle’ we’re left with an annulus. But, of course, an annulus is simple a disk with one hole, or a sphere with two holes. In other words, we again get a cylinder. In all three cases we don’t get a map. For a map, every face must be homeomorphic to a disk.

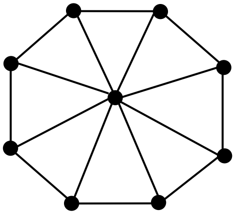
A graph can be **mapped onto** a surface if it can be embedded in such a way that, if the edges are removed, the surface falls apart into open topological disks (open because they have no boundary). We call these disks, with their boundaries restored, the **faces**. So, in fact although $\mathbf{K}_3$ can be embedded in a torus, it cannot be 2-cell embedded. While graphs can be too complicated to be 2-cell embedded in a certain surface, they can also be too simple.

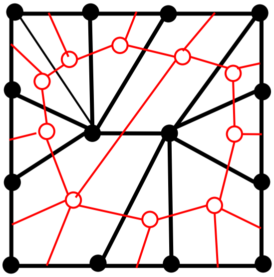
Clearly, if a graph can be 2-cell embedded in a surface it can be embedded in the surface.

If we have a map on a surface with no unidentified edges, we can construct its **dual map**. This is the map where we place a vertex in the middle of each face and draw an edge to the adjacent faces across each edge on their boundary.

Example 12: The following two maps on a sphere are duals of one another. (Don't forget the outside face. The middle vertex in each case represents the outside face of the other map.)







Example 13: These two maps on a Torus are duals of one another. One Has solid black vertices and heavy Black edges. The other has hollow

Red vertices and red edges. Note that unlike the previous example there is no outside face.

Example 14: $\mathbf{K}_4$ is planar and so can be embedded in a torus, but $\mathbf{K}_4$ cannot be mapped onto a torus.

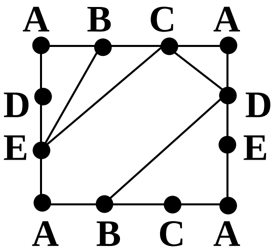
For $\mathbf{K}_4$ we have V = 4, E = 6 and therefore on a torus (χ = 0) F would have to be 2. Now $\frac{2E}{F} = \frac{12}{2}$ = 6. Now although this is less than the girth, it means we must have one face with more vertices than there are altogether.

Example 15: $\varepsilon(K_3, \mathbf{T}) = 1$ but $\mu(K_3, \mathbf{T}) = 0$. That is, $\mathbf{K}_3$ can be embedded in a torus but cannot be mapped onto a torus.

Theorem 1: Let G be a graph and let H be a subgraph of G. Let S, T be any surfaces S, T. Then:
(1) If G can be mapped onto S it can be embedded in S.
(2) G can be mapped onto a sphere if and only if it can be embedded on a sphere.
(3) G can be embedded in S + $\mathbf{D}$ if and only if it can be embedded in S.
(4) G can be mapped onto S + $\mathbf{D}$ if and only if it can be mapped onto S.
(5) If G can be embedded in S then it can be embedded in S + T.

Proof: Left as an exercise.

Example 16: K_5 can be mapped onto a torus.



We need to examine the faces carefully to ensure that they are all homeomorphic to disks.

An embedding is certainly the more interesting concept. After all, if we wanted to connect people by telephone in a 2-dimensional world that lay on a torus, it wouldn't matter what the faces were homeomorphic to. However mappings of a graph are easier to prove theorems about.

§5.5. Deciding Whether a Graph Can Be Mapped Onto a Surface.

Theorem 2 (COOPER): Suppose a graph has E edges and V vertices and the girth is g. Suppose G can be mapped onto a surface S is a surface with no holes and with Euler characteristic χ. Let $F = \chi + E - V$. Then:

$$V \geq \frac{2E}{F} \geq g.$$

Proof: The map will have F faces where F = χ + E – V. The quantity $\frac{2E}{F}$ will be the average number of edges per face. The map must therefore have a face with at least this number of edges. But if this exceeds the number of vertices that face must have a repeated vertex and hence must be bounded by an identified edge, contradicting the fact that the face is homeomorphic to a disk.

On the other hand if $\frac{2E}{F}$ is less than the girth the map must have a face with fewer edges than the girth, again a contradiction.

Procedure for finding out if a graph, G, can be mapped onto a surface, S.

(1) In G, remove any vertices of degree 0. Remove any vertices of degree 1 and the associated edge. Remove any vertex of degree 2 by combining the two edges at that vertex into a single edge. Repeat this until the degree of all vertices is at least 3.

(2) If the surface has boundaries, remove them and use the corresponding surface with no boundaries.

(3) Count the number of vertices, V, and edges E. Find the Euler characteristic, χ, of S. Find the girth, *g*, of G.

(4) Compute F = χ + E – V.

(5) If F < 2 the graph cannot be mapped onto the surface.

If F ≥ 2, continue.

(6) Compute $\frac{2E}{F}$.

(7) If $\frac{2E}{F} > V$ or $\frac{2E}{F} < g$ then G cannot be mapped onto S. But if $g \le \frac{2E}{F} \le V$ then it *may* be possible to map G onto S, but this is no proof that it can be. The only way I know of that guarantees that a graph can be mapped onto a surface is to actually do it. And, of course the most convenient way to do this is to represent S as a polygon with identified edges.

It would be nice to be able to prove that a graph can't be *embedded* in a surface, as distinct from merely being able to map it. I don't know any methods for doing this. However I strongly suspect that if $F \ge 2$ and $\frac{2E}{F} < g$ then the graph cannot be embedded in the surface.

Theorem 3: The largest value of n for which $\mathbf{K}_n$ can be mapped onto a torus is 7.
Solution: For $\mathbf{K}_n$, $V = n$ and $E = \frac{1}{2} n(n-1)$.
Suppose that $\mathbf{K}_n$ can be mapped onto a torus.
The resulting map will have F faces where:

$$n + F - \tfrac{1}{2} n(n-1) = 0.$$

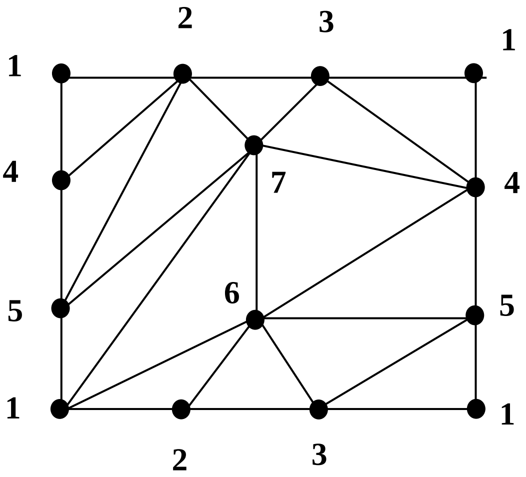
So $F = \dfrac{n^2 - 3n}{2}$. $\dfrac{2E}{F} = \dfrac{2n(n-1)}{n^2 - 3n}$.

The girth of $\mathbf{K}_n$ is 3.

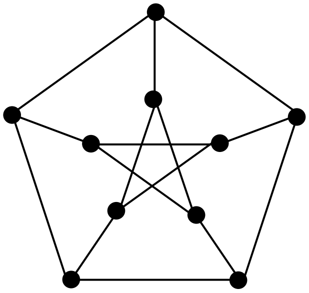
So $\dfrac{2n(n-1)}{n^2 - 3n} \ge 3$, in which case $3n^2 - 9n \le 2n^2 - 2n$.

Hence $n^2 - 7n \leq 0$. Since $n > 0$ we may divide this inequality by n to obtain $n \leq 7$.

We now show that $\mathbf{K}_7$ can be mapped onto a torus.



Example 17: The following graph is known as the Petersen Graph:



$V = 10$
$E = 15$
If it can be mapped onto a sphere:
$F = 2 + 15 - 10 = 7$

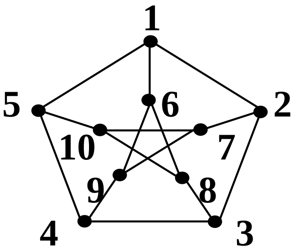
$$\frac{2E}{F} = \frac{30}{7} < 5$$

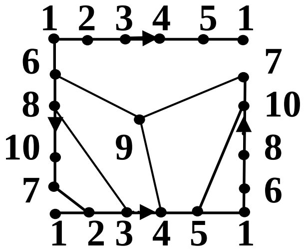
$g = 5$
So the Petersen graph can't be mapped onto a sphere. In other words it isn't planar.

For a Klein bottle, $\chi = 0$. So now $F = 0 + 15 - 10 = 5$ and $\frac{2E}{F} = \frac{30}{5} = 6 > g$.

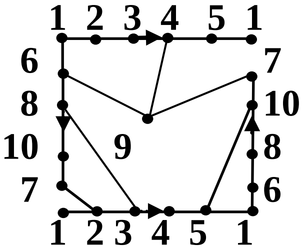
The $\frac{2E}{F} < g$ test fails. So it might be possible to map the Petersen graph onto a Klein bottle. On the other hand it might be impossible, which isn’t much help!

But in fact we *can* embed the Petersen graph onto a Klein bottle as follows:





However this is not map because the top face is surrounded by two cycles 1-2-3-4-5-1 and 1-2-7-1. But we only need to change one edge to obtain a mapping:



The face in the top-right-hand corner is homeomorphic to a disk.

Whenever you display such an embedding it's important to do what we've done here – label the vertices of the original graph and the map so that it's easy to check that it's indeed the same graph. (Vertex 1 is adjacent to 2 in both graphs, it's not adjacent to vertex 3 in either graph, and so on.)

Theorem 4: If $\mathbf{K}_n$ can be mapped onto a sum of m toruses then $n \le \dfrac{7 + \sqrt{48m + 1}}{2}$.

Proof: Suppose $n \ge 3$ and that $\mathbf{K}_n$ can be mapped onto a sum of n toruses.

Then for $\mathbf{K}_n$, $V = n$, $E = ½\, n(n - 1)$ and the girth is $g = 3$.

The Euler characteristic of the sum of m toruses is $2 - 2m$.

So $F = E + \chi - V$

$$= ½\,[n^2 - 3n - 4m + 4].$$

Since $\mathbf{K}_n$ can be mapped onto m**T**:

$$\frac{2E}{F} = \frac{2n(n-1)}{n^2 - 3n - 4m + 4} \ge 3.$$

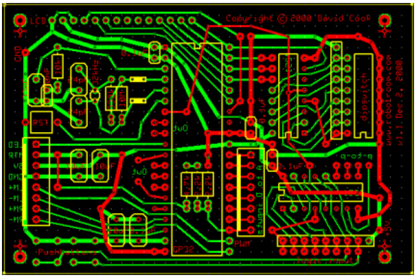
That is, $n^2 - 7n - 12(m - 1) \le 0$.

The zeros of $n^2 - 7n - 12(m - 1)$ are $\dfrac{7 \pm \sqrt{48m + 1}}{2}$ and so n cannot exceed the larger zero.

Hence $n \le \dfrac{7 + \sqrt{48m + 1}}{2}$.

§5.6. Printed Circuit Boards.

A printed circuit board has electronic components laid out on both sides of a board with the connecting tracks ‘printed’ on the board. They can be considered as graphs where the vertices occur on both sides of the surface and the edges on each side form a planar graph. We insist that each edge lies entirely on one side or the other. A graph is **2-planar** if it is the union of two subgraphs, G_1 and G_2, on the same set of vertices, where G_i and G_2 are both planar.



Theorem 5 (COOPER): If $\mathbf{K}_n$ is 2-embeddable in a plane then $n \leq 10$.

Proof: Suppose that $\mathbf{K}_n$ can be 2-embedded on a sphere (the corresponding surface with no edges) and let the planar graphs on the two sides be G_1 and G_2. The number of vertices in each of these subgraphs will be n.

Suppose that the number of edges in G_1 and G_2 are E_1 and E_2 respectively and suppose that, when embedded in the sphere, the number of faces are F_1 and F_2.

From Euler’s formula, $F_i = E_i - n + 2$ for $i = 1, 2$. Since each subgraph must be planar we must have

$$\frac{2E_i}{F_i} \geq 3, \text{ for each } i.$$

Hence, for each i, $2E_i \geq 3F_i = 3(E_i - n + 2)$ and so

$$E_i \leq 3n - 6.$$

Since every vertex must be connected to every other, on one side or the other, we have $E_1 + E_2 = \frac{1}{2} n(n - 1)$.

Hence $\frac{1}{2} n(n - 1) \leq 6n - 12$.

It follows that $n^2 - 13n + 24 \leq 0$.

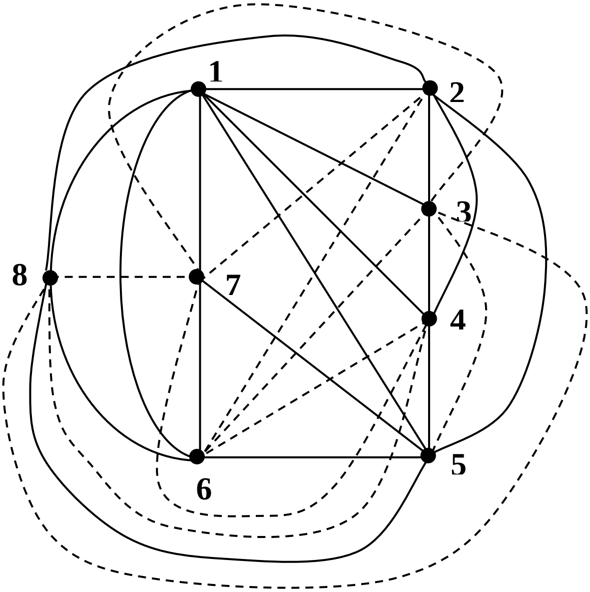
The zeros of $n^2 - 13n + 24$ are $\dfrac{13 \pm \sqrt{169 - 96}}{2}$ and so

$$n \leq \frac{13 + \sqrt{73}}{2} = 10.77\ldots$$

Since n is an integer, $n \leq 10$.

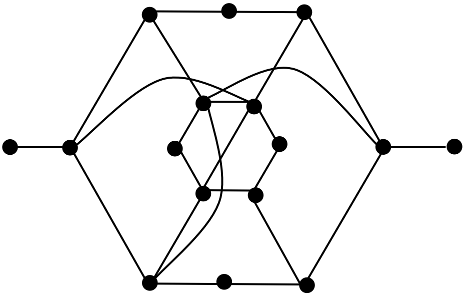
Corollary: The largest value of n for which $\mathbf{K}_n$ can be 2-embedded in a plane is 8, 9 or 10.

Proof: The following is a 2-embedding of $\mathbf{K}_8$ in a plane. Is it possible to 2-embed $\mathbf{K}_9$ or even $\mathbf{K}_{10}$?



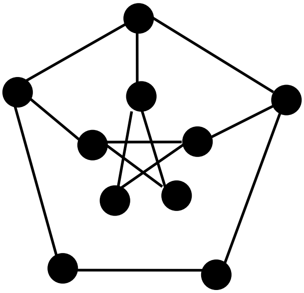
EXERCISES FOR CHAPTER 5

Exercise 1:
Show that the following graph can be mapped onto a projective plane, but not onto a disk.

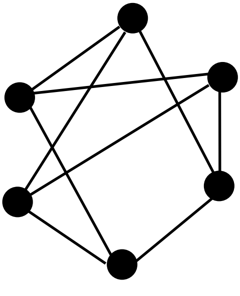


Exercise 2:
Determine (with reasons) which of the following graphs are planar, which can be mapped onto a Möbius Band and which can be mapped onto a torus.

(a)

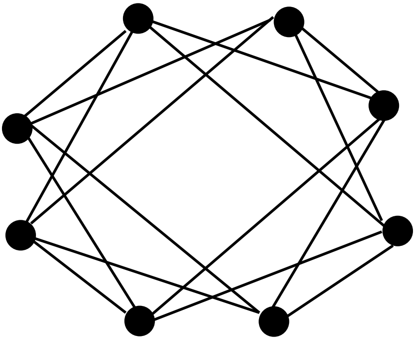


(b)



Exercise 3:

(a) What is the girth of the following graph?



(b) Prove that this graph is not planar.

(c) Can this graph be mapped onto a projective plane?

(d) Can this graph be mapped onto a torus?

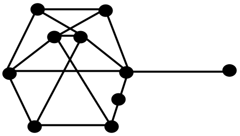
(e) Can this graph be mapped onto a Möbius Band?

Give reasons for your answers to (c) to (e).

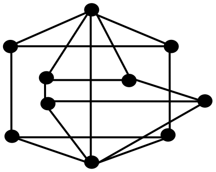
Exercise 4:

For each of the following graphs determine whether or not it can be mapped onto a cylinder and whether or not it can be mapped onto a torus with 3 holes.

Graph A



Graph B



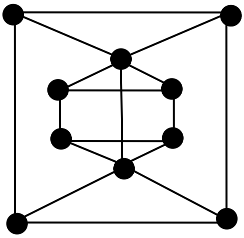
Exercise 5:

(i) Draw a diagram for $\mathbf{K}_{3,4}$, as a graph on 7 vertices. Draw a square with identified edges that represents the Klein bottle.

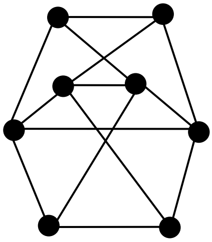
(ii) Show that $\mathbf{K}_{3,4}$ can be mapped onto a Klein bottle.

Exercise 6:

For each of the following graphs determine whether or not it can be mapped onto:
(i) a cylinder; (ii) a torus.



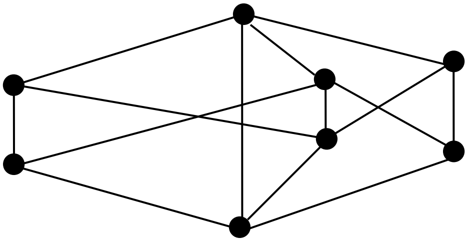
Graph A



Graph B

Exercise 7:

(a) Suppose the following graph is planar. Calculate the average number of edges per face in the resulting map. Explain how this leads to a contradiction.



(b) Can the graph in (b) be mapped onto a cylinder (Give reasons.)

Exercise 8:

(a) Show that if $\mathbf{K}_n$ can be mapped onto a surface with no boundaries and Euler characteristic χ then

$$n \leq \frac{7 + \sqrt{49 - 24\chi}}{2}.$$

(b) Hence find the largest value of n for which $\mathbf{K}_n$ can be mapped onto the connected sum of a torus, a projective plane and a disk.

Exercise 9:

(a) Draw pictures of $\mathbf{K}_{mn}$ for all m, n with $m + n = 8$.

(b) Prove that if K_{mn} can be mapped onto a surface of weight w then, unless $m = n = 1$, $w \geq \frac{1}{2}(m - 2)(n - 2)$.

(c) Find all values of m, n for which $\mathbf{K}_{mn}$ is planar. (Illustrate these cases with suitable picture.)

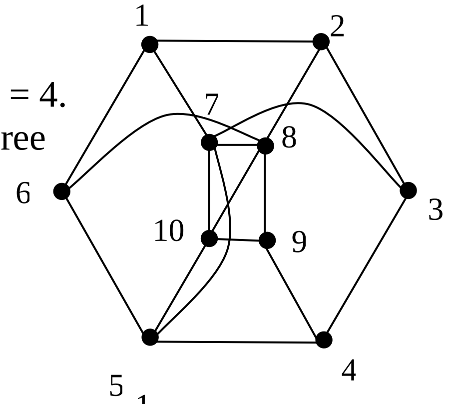
(d) Find all values of m, n for which $\mathbf{K}_{mn}$ is not planar but can mapped onto a torus. (Illustrate each of these with a suitable picture, using a polygon with identified edges to represent the torus.)

Exercise 10: Prove that if $\mathbf{K}_n$ can be mapped onto a torus then $n \leq 13$.

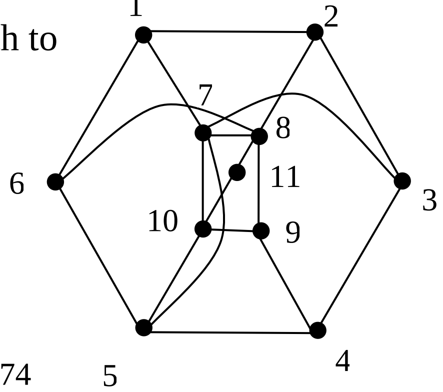
Exercise 11: Can K_4 be mapped onto a torus?

SOLUTIONS FOR CHAPTER 5

Exercise 1:
The given graph has girth $g = 4$. Deleting the vertices of degree 1 and 2 we obtain the following graph:



But this has reduced the girth to 3, not $g = 4$ as previously. So insert back a vertex to turn the triangles into quadrilaterals, as in the following diagram.



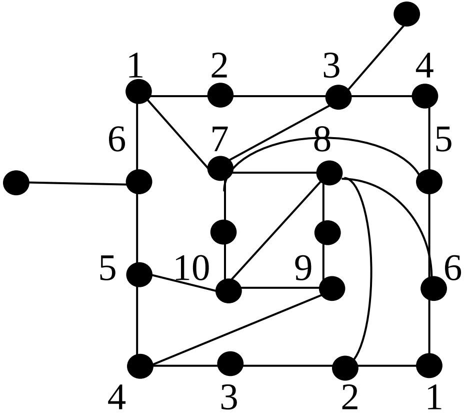
Clearly if the original graph is embeddable in a surface, then so can both of these be embedded in the same surface, and conversely.

Now this new graph has V = 11 and E = 19, so for an embedding in a sphere ($\chi = 2$) we would need

$$F = 2 + 19 - 11 = 10 \text{ faces},$$

giving $\frac{2E}{F} = \frac{38}{10} = 3.8 < 4 = g$. Being less than the girth means that the graph is not embeddable in a sphere. Hence the original graph is not embeddable in a sphere.

With the surface being a Projective Plane ($\chi = 1$) there need be only F = 1 + 19 – 11 = 9 faces, which then gives $\frac{2E}{F} = \frac{38}{9} = 4\frac{2}{9} > 4 = g$. Now an embedding could be possible. Indeed it as the following diagram shows.



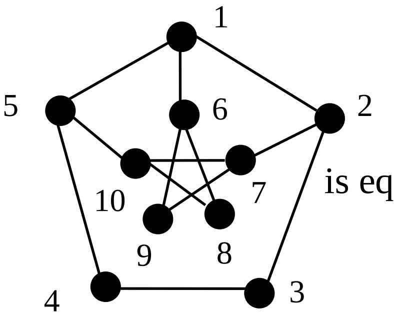
Exercise 2:

(a) By inspection the girth = 5. The number of edges, E, is 13 and the number of vertices, V, is 10.

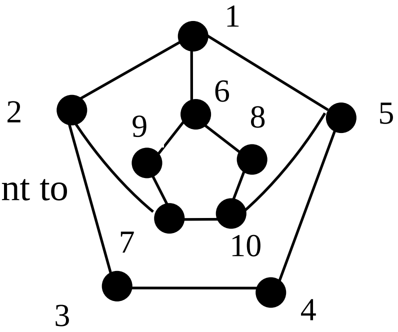
A graph is planar if and only if it can be embedded in a sphere. Suppose it is planar.

By Euler's formula V + F – E = 2 and so F = 5.

The average number of edges per face is $\frac{2E}{F} = \frac{26}{5} > 5$ so it **might** be planar. Let's try.

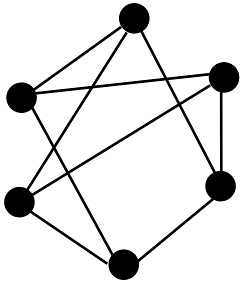


is equivalent to



and hence the graph is indeed planar. Hence it is embeddable in a Möbius Band and a torus.

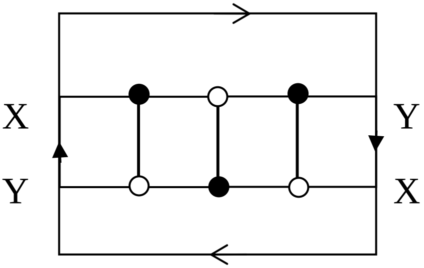
(b) This graph is $\mathbf{K}_{3,3}$ which we know is non-planar, but can be mapped onto both the Projective Plane and the torus.



We have V = 6 and E = 9 and the girth is $g = 4$. For the Projective Plane, with $\chi = 1$, there would need to be

$$F = E + \chi - V = 9 + 1 - 6 = 4 \text{ faces.}$$

Now $\frac{2E}{F} = \frac{18}{4} > 4 = g$, so it might be possible to map it onto a Projective Plane. The following diagram shows it as a map on the Projective Plane.

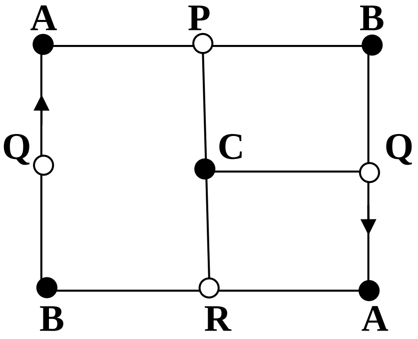


Note that if the block arrows ➔ had been reversed this would not have worked.

Since a Möbius Band is a Projective Plane with one hole, it can be mapped onto a Möbius Band if and only if it can be mapped onto a Projective Plane.
For a Projective Plane, $\chi = 1$, $g = 4$, V = 6, E = 9, so if it can be mapped onto a Projective Plane,

$$F = E + 1 - V = 4. \ \frac{2E}{F} = \frac{18}{4} > 4.$$

The test is inconclusive, so it might be possible for it to be mapped onto a Projective Plane, or equivalently, onto a Möbius Band. The following diagram shows that it can be mapped onto a Möbius Band.



This can easily be adapted to an embedding in a torus.

Exercise 3: This graph is $\mathbf{K}_{4,4}$.

(a) girth = 4

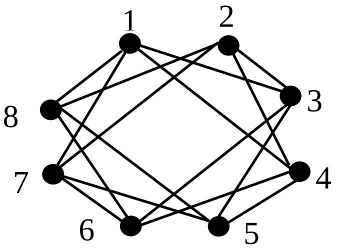
(b) V = 8, E = 16 (rather than count them all, simply notice that each vertex has degree 4 (4 edges to each vertex). Multiplying by the number of vertices, and dividing by 2 (since each edge connects two vertices) we get 16.

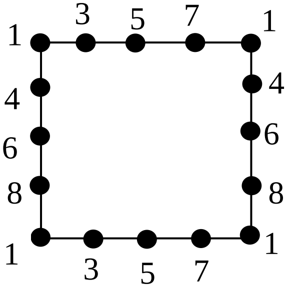
If the graph is planar it can be embedded in a sphere, and the number of faces that would result would be F = E + 2 – V = 10. $\frac{2E}{F} = \frac{32}{10} < 4$, so we get a contradiction. This graph is not planar.

(c) If it could be mapped onto a projective plane, F = 9 and $\frac{2E}{F} = \frac{32}{9} < 4$. So it cannot be mapped onto a projective plane.

(d) For the torus, F = 8 and $\frac{2E}{F} = 4$. The test fails. Perhaps it can be mapped onto a torus. If it can't it will need another argument to prove this. So we attempt to draw it on a torus.

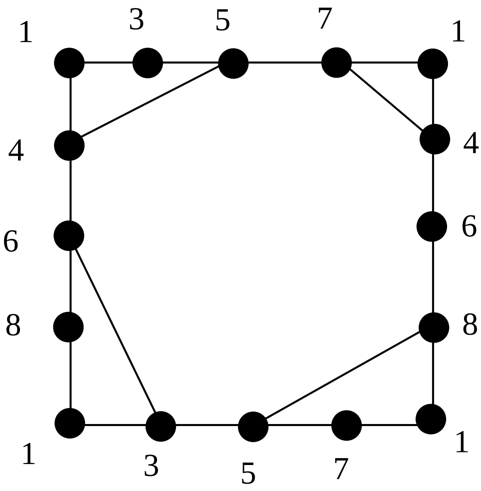
It's a good idea to locate one point at the corners of the diagram (all identified) and some others on the edges. We don't have to use the edges on the diagram made up of edges in the graph but it makes life simpler if we do. So let's put '1' at the four corners and look for a couple of cycles of edges that intersect only in that vertex. There's a cycle 1-3-5-7-1 and 1-4-6-8-1. So we plot these points along the edges in appropriate places.



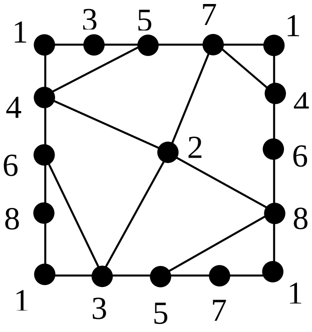


There's still vertex 2 to locate. It will go somewhere in the middle of the square. We'll ignore it for the moment. Now we put in the remaining edges, except for those having 2 as an endpoint. Because the other vertices have two locations we have choices as to which locations we use.

We make choices, with an eye to allowing '2' to be able to access the four vertices it will need to be joined to.



Clearly, if we locate 2 in the largest face in this diagram it will be able to reach 3, 4, 7 and 8.



Exercise 4:

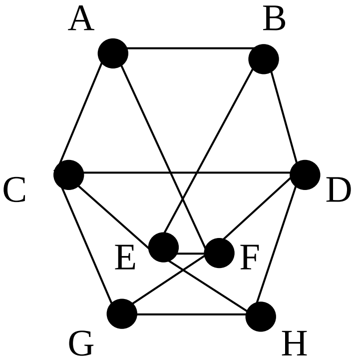
Graph A:

girth = 4

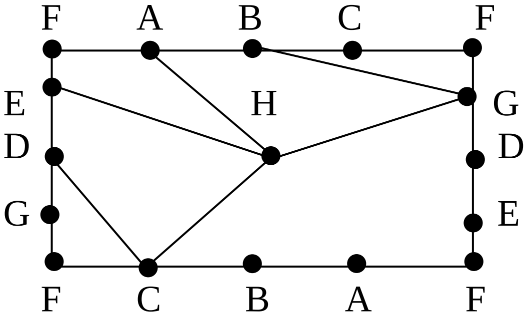
V = 8, E = 14. Suppose graph A can be embedded in a sphere. Let the number of faces be F.

Then F = 2 + 14 – 8 = 8. Hence $\frac{2E}{F} = \frac{28}{8}$ < girth.

So graph A cannot be embedded in a sphere.

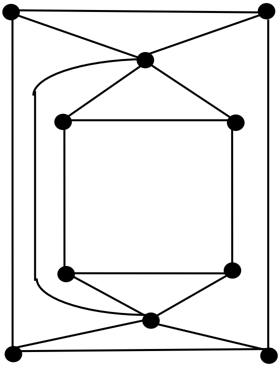


It can be mapped onto a projective plane as follows:



Graph B:

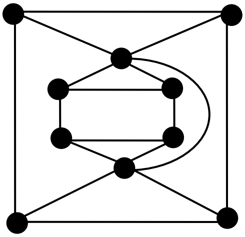
The graph can be redrawn as:



so it is planar. Hence it can be mapped onto any surface, in particular the sphere and the Möbius Band.

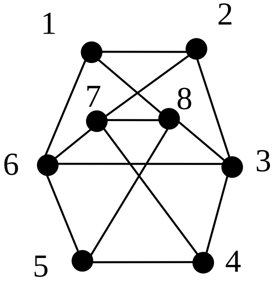
Exercise 6:

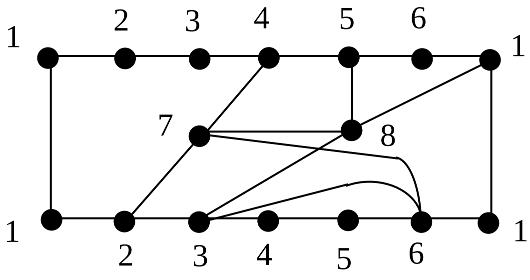
Graph A can be drawn as:



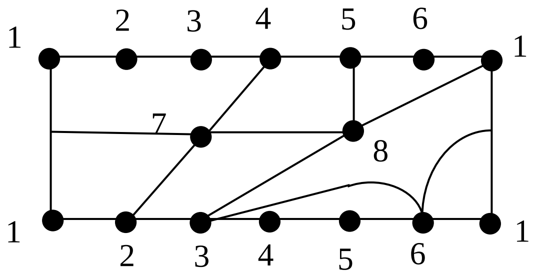
so it is embeddable in the disk and hence in all surfaces, including the cylinder and the torus.

For graph B, V = 8, E = 14 and if embedded in the sphere the number of faces would be F = 14 + 2 – 8 = 8. So $f = \frac{2E}{F} = \frac{28}{8} = 3.5 < 4$ = girth. Hence graph B cannot be embedded in the sphere and hence not in the cylinder. But we can almost map it onto a torus:





There’s just one crossing left. But we can join 6 to 7 by going up and to the right to the middle of the vertical and reappear at the corresponding point on the left (since the rectangle represents a torus).



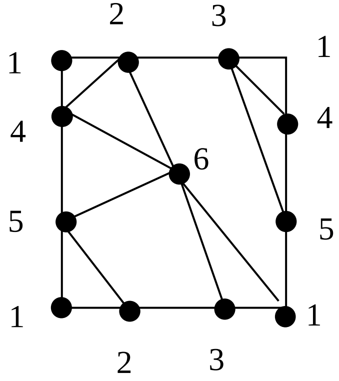
This would also work on a Klein Bottle.

Exercise 7: (a) V = 8, E = 14. If embedded in a sphere the resulting map would have

F = 14 + 2– 8 = 8 faces and so the average number of edges per face would be *f* = 28/8 = 3½.

Since this is less than the girth (which is 4), we get a contradiction. Hence this graph cannot be embedded in a sphere.

(b) Since the cylinder is a sphere with two holes and since holes do not affect embeddability, a surface is embeddable in a cylinder if and only if it is embeddable in a sphere. Hence the above graph is not embeddable in a cylinder. However it can be mapped onto a torus.



Exercise 8:

(a) The number of vertices of $\mathbf{K}_n$ is V = n and the number of edges is E = ½ n(n – 1).

Its girth is $g = 3$.

Suppose $\mathbf{K}_n$ can be mapped onto S. The number of faces would be $F = E - V - \chi$

$$= \tfrac{1}{2}\,(n^2 - 3n + 2\chi).$$

Then $\dfrac{2E}{F} = \dfrac{2n^2 - 2n}{n^2 - 3n + 2\chi}$.

Hence $\dfrac{2n^2 - 2n}{n^2 - 3n + 2\chi} \geq 3$ and so $n^2 - 7n + 6\chi \leq 0$.

Hence $n \leq \dfrac{7 + \sqrt{49 - 24\chi}}{2}$.

(b) $\chi_0(\mathbf{T} + \mathbf{P}) = 2 + 1 = 3$ so $\chi(\mathbf{T} + \mathbf{P}) = 2 - 3 = -1$ so by part (a), if $\mathbf{K}_n$ can be embedded in $\mathbf{T} + \mathbf{P}$ then

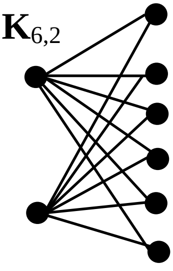
$$n \leq \frac{7 + \sqrt{73}}{2} < 8.$$

So $\mathbf{K}_n$ cannot be embedded in $\mathbf{T} + \mathbf{P}$ if $n \geq 8$.

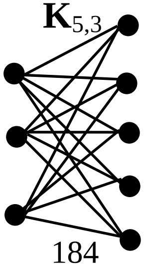
But $\mathbf{K}_7$ can be embedded in a torus and hence can be embedded in $\mathbf{T} + \mathbf{P}$ so the largest value of n for which $\mathbf{K}_n$ can be embedded in $\mathbf{T} + \mathbf{P}$ is $\boldsymbol{n = 7}$.

Exercise 9: (a)

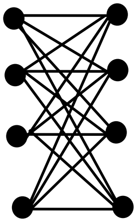
$\mathbf{K}_{6,2}$



$\mathbf{K}_{5,3}$



$\mathbf{K}_{4,4}$



(b) Suppose $\mathbf{K}_{m,n}$ can be mapped onto a surface (with no holes) of weight W. The Euler characteristic of this surface is $\chi = 2 - W$.

Case I: $n \geq 2$. Here $g = 4$, $V = m + n$, $E = mn$.
Thus $F = (2 - W) + mn - m - n$ and so

$$f = \frac{2mn}{mn - m - n + 2 - W} \geq 4.$$

Thus $4mn - 4m - 4n + 8 - 4W \leq 2mn$ and so
$4W \geq 2mn - 4m - 4n + 8 = 2(m - 2)(n - 2)$ and hence

$$W \geq \tfrac{1}{2}(m - 2)(n - 2).$$

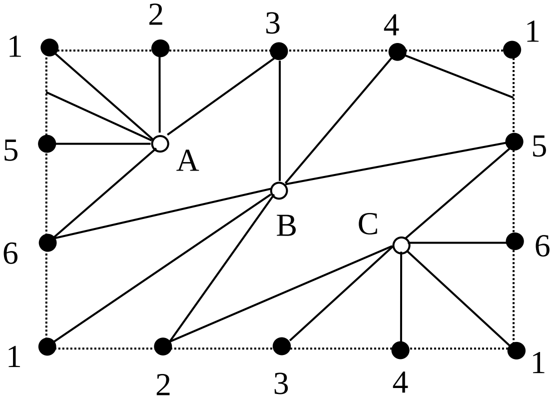
Case II: $n = 1$, $m > 1$. Then $\tfrac{1}{2}(m - 2)(n - 2) \leq 0 \leq W$.

(c) If $n = 1$, $\mathbf{K}_{m,n}$ is embeddable in a plane (and hence a torus).

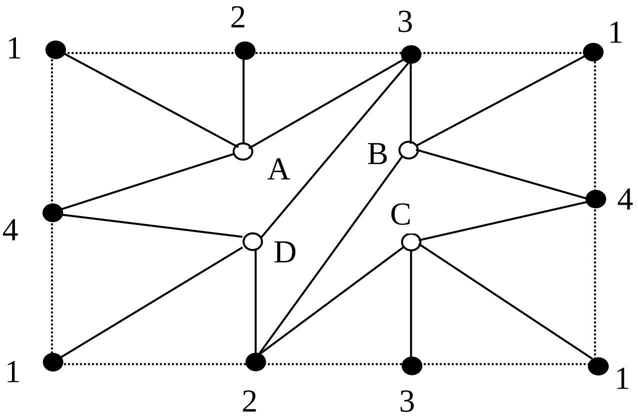
Suppose $n \geq 2$ and that $\mathbf{K}_{m,n}$ can be mapped onto a torus (with weight $W = 2$).
From (b) $(m - 2)(n - 2) \leq 4$.
The only solutions of this inequality for positive integers m, n with $m \geq n \geq 2$ are:
$n = 2$, $m =$ any value.
(These are planar so can be ignored);
$n = 3$, $m = 3$;
$n = 3$, $m = 4$;
$n = 3$, $m = 5$;
$n = 3$, $m = 6$;
$n = 4$, $m = 4$.

The last five all include **K**$_{3,3}$ which is not planar so none of these is planar.

It remains to prove that they are can be mapped onto a torus. K$_{6,3}$ can be embedded in a torus and hence so can K$_{5,3}$ and K$_{4,3}$ and K$_{3,3}$.



That just leaves **K**$_{4,4}$.



Exercise 10: Suppose that $\mathbf{K}_n$ can be mapped onto a hollow torus and let the planar graphs on the two sides be G_1 and G_2. The number of vertices in each of these subgraphs will be n.

Suppose that the number of edges in G_1 and G_2 are E_1 and E_2 respectively and suppose that, when embedded in the sphere, the number of faces are F_1 and F_2.

From Euler's formula, $F_i = E_i - n$ for $i = 1, 2$ since

$$\chi(\text{torus}) = 2.$$

Since each subgraph must be planar we must have

$$\frac{2E_i}{F_i} \geq 3, \text{ for each } i.$$

Since every vertex must be connected to every other, on one side or the other, we have

Hence, for each i, $2E_i \geq 3F_i = 3(E_i - n)$ and so $E_i \leq 3n$.

Now for $\mathbf{K}_n$, $E_1 + E_2 = \frac{1}{2} n(n - 1)$.

Hence $\frac{1}{2} n(n - 1) \leq 6n$. It follows that $n^2 - 13n \leq 0$ and so $n \leq 13$.

Exercise 11:

Suppose $\mathbf{K}_4$ can be mapped onto a torus. Now $\mathbf{K}_4$ has $V = 4$ vertices and $E = 6$ edges.

Suppose that a resulting map has F faces.

Then $4 + F - 6 = 2$ so $F = 2$.

Now $\frac{2E}{F} = \frac{12}{2} = 6 >$ girth $= 3$. Hence the $\frac{2E}{F}$ test is inconclusive. But if the average number of edges per face is 6 there must be a face with at least 6 edges and this would require at least 6 vertices. But there are only 4 vertices. So $\mathbf{K}_4$ cannot be mapped onto a torus.

Alternatively we could argue that, since there would be only 2 faces, each homeomorphic to **D**, if we cut around one of the faces this would leave a hole in the torus. Hence the other face would have to be **D** + **T**, not **D**.